CHAPTER

15 Probability

Mutually Exclusive Events

A set of events is said to be mutually exclusive if occurrence of one of them precludes the occurrence of any of the remaining events.

Thus, $E_1, E_2, ..., E_n$ are mutually exclusive if and only if $E_i \cap E_j = \phi$ for $i \neq j$.

Independent Events

Two events are said to be independent, if the occurrence of one does not depend on the occurrence of the other.

For example, when a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.

Complement of An Event

The complement of an event E, denoted by $\overline{E}$ or E' or E^c, is the set of all sample points of the space other than the sample points in E.

For example, when a die is thrown, sample space

$S = \{1, 2, 3, 4, 5, 6\}$.

If $E = \{1, 2, 3, 4\}$, then $\overline{E} = \{5, 6\}$.

Note that $E \cup \overline{E} = S$.

Mutually Exclusive and Exhaustive Events

A set of events $E_1, E_2, ..., E_n$ of a sample space S form a mutually exclusive and exhaustive system of events, if

(*i*) $E_i \cap E_j = \phi$ for $i \neq j$ and

(*ii*) $E_1 \cup E_2 \cup ... \cup E_n = S$

Notes:

(*i*) $O \leq P(E) \leq 1$, i.e. the probability of occurrence of an event is a number lying between 0 and 1.

(*ii*) $P(\phi) = 0$, i.e. probability of occurrence of an impossible event is 0.

(*iii*) $P(S) = 1$, i.e. probability of occurrence of a sure event is 1.

ODDs in Favour of an Event and ODDs Against An Event

If the number of ways in which an event can occur be m and the number of ways in which it does does not occur be n, then

(*i*) Odds in favour of the event = $\frac{m}{n}$ and

(*ii*) Odds against the event = $\frac{n}{m}$.

Some Important Results on Probability

1. $P(\overline{A}) = 1 - P(A)$.
2. If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
3. If A and B are mutually exclusive events, then $A \cap B = \phi$ and hence $P(A \cap B) = 0$.
   $\therefore P(A \cup B) = P(A) + P(B)$.
4. If A, B, C are any three events, then $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.
5. If A, B, C are mutually exclusive events, then $A \cap B = \phi$, $B \cap C = \phi$, $C \cap A = \phi$, $A \cap B \cap C = \phi$ and hence $P(A \cap B) = 0$, $P(B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$.
   $\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$.
6. $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$.
7. $P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B)$.
8. $P(A) = P(A \cap B) + P(A \cap \overline{B})$.
9. $P(B) = P(B \cap A) + P(B \cap \overline{A})$.
10. If $A_1, A_2, ..., A_n$ are independent events, then $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \cdot P(A_2) ... P(A_n)$.
11. If $A_1, A_2, ..., A_n$ are mutually exclusive events, then
    $P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n)$.
12. If $A_1, A_2, ..., A_n$ are exhaustive events, then $P(A_1 \cup A_2 \cup ... \cup A_n) = 1$.
13. If $A_1, A_2, ..., A_n$ are mutually exclusive and exhaustive events, then
    $P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n) = 1$.
14. If $A_1, A_2, ..., A_n$ are n events, then
    (*i*) $P(A_1 \cup A_2 \cup ... \cup A_n) \leq P(A_1) + P(A_2) + ... + P(A_n)$.
    (*ii*) $P(A_1 \cap A_2 \cap ... \cap A_n) \geq 1 - P(\overline{A}_1) - P(\overline{A}_2) ... - P(\overline{A}_n)$.

Conditional Probability

$P\left(\frac{B}{A}\right)$ = Probability of occurrence of A, given that B has already happened.

$= \dfrac{P(A \cap B)}{P(B)} = \dfrac{n(A \cap B)}{n(B)}$

1. **Multiplication theorems on probability**

   (*i*) If A and B are two events associated with a random experiment, then $P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right)$, If $P(A) \neq 0$ or $P(A \cap B) = P(B) \cdot P\left(\frac{B}{A}\right)$, if $P(B) \neq 0$

   (*ii*) **Multiplication theorems for independent events:** If A and B are independent events associated with a random experiment, then $P(A \cap B) = P(A) \cdot P(B)$ i.e. the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities. By multiplication theorem, we have $P(A \cap B) = P(A) \cdot P(B/A)$. Since A and B are independent events, therefore

   $P(B/A) = P(B)$. Hence, $P(A \cap B) = P(A) \cdot P(B)$.

2. **Probability of at least one of the n independent events:** If $p_1, p_2, p_3, \ldots p_n$ be the probabilities of happening of n independent events $A_1, A_2, A_3, \ldots A_n$ respectively, then

   (*i*) Probability of happening none of them = $P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \ldots \cap \overline{A}_n) = P(\overline{A}_1) \cdot P(\overline{A}_2) \cdot P(\overline{A}_3) \ldots P(\overline{A}_n)$
   $= (1 - p_1)(1 - p_2)(1 - p_3) \ldots (1 - p_n)$.

   (*ii*) Probability of happening at least one of them
   $= P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n) = 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3) \ldots P(\overline{A}_n)$
   $= 1 - (1 - p_1)(1 - p_2)(1 - p_3) \ldots (1 - p_n)$

Law of Total Probability

$$P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + \ldots + P(E_n) \cdot P\left(\frac{A}{E_n}\right)$$

Baye's rule as $P(E_i/A) = \dfrac{P(E_i) \cdot P(A/E_i)}{\sum_{k=1}^{n} P(E_k)\, P(A/E_k)}$.

Binomial Distribution

The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

Random Variable

The expectation (mean) of the random variable X is defined as $E(X) = \sum_{i=1}^{n} p_i x_i$ and the variance of X is defined as

$$\text{var}(X) = \sum_{i=1}^{n} p_i (x_i - E(X))^2 = \sum_{i=1}^{n} p_i x_i^2 - (E(X))^2 .$$